EIGENFUNCTIONS ON A STADIUM ASSOCIATED
WITH AVOIDED CROSSINGS OF ENERGY LEVELS

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1. INTRODUCTION

The most primitive form of a stadium is a circle, say of radius r. If we separate the left and right halves of the circle slightly and connect these two semi circles at the top and bottom by straight lines we have a classical stadium (racetrack). Let a be the horizontal distance from the center of the figure to either the left or right semi circle; that is, the distance of each 'straightaway' is $2a$.

By the 1970's researchers were using numerical studies to examine the stochastic behavior of quantum mechanical systems corresponding to chaotic and quasi-periodic behavior in classical mechanics. In an unpublished letter written at the Oak Ridge National Laboratories in the early 1980's, [1], the authors studied a Hamiltonian as a free particle confined in a stadion (stadium) boundary. They developed a FORTRAN code to compute the 10 smallest eigenvalues of even-even symmetry for the Schroedinger equation for this system and plotted (by hand!) line graphs of the eigenvalues versus a/r for $r = 1$. The correlation plot showed many avoided crossings. Word was that Wigner proved these graphs could not cross when the corresponding eigenfunctions had the same symmetry; but after the intervening years a firm reference evades us. Suggested earlier references include [2] - [6], and [7] for a more recent paper. Computations in [1] were done only for the upper right quadrant of the stadium and the symmetry was forced.

The present paper is based on the earlier work at Oak Ridge. We have used a slightly updated version of the old FORTRAN code, this time on the entire region, again with $r = 1$. It is our purpose to indicate certain striking properties of eigenfunctions near avoided crossings. Plots for these eigenfunctions were not made at the time [1] was written, nor were the data retained.

2. The eigenvalue problem

We now consider the eigenvalue problem

$$-u_{11} - u_{22} = \lambda u$$

on the interior of the stadium where u is 0 on the boundary and λ represents energy. For each stadium configuration determined by the separation distance a the code picks out a designated number of eigenvalues λ and solves for the values of the associated eigenfunctions u.
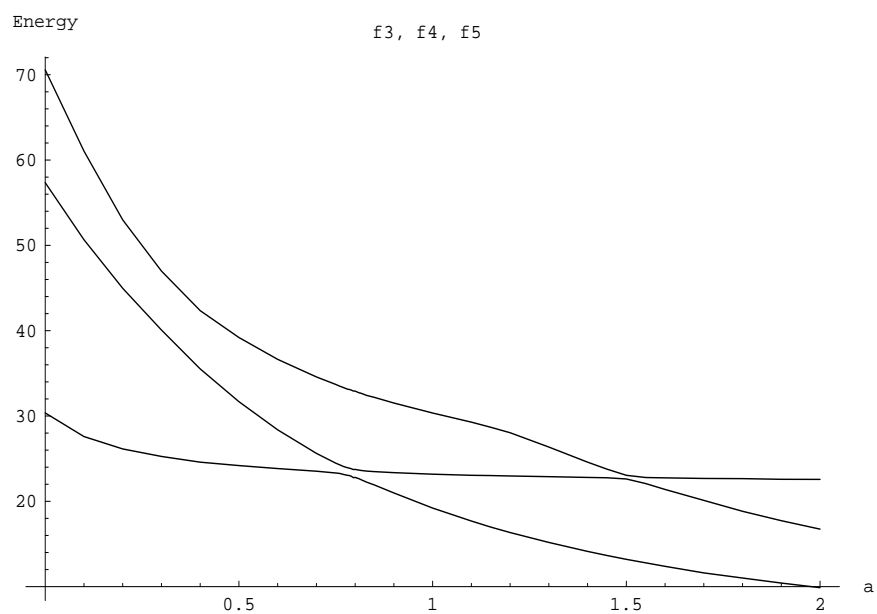
Mathematica was used to make surface and contour plots of the eigen functions. The contour plots provided especially accessible information so we concentrated on those. In fact, we visually selected eigenfunctions that produced contour graphs that were 'even-even', symmetric about both the x and y axes. Then we plotted line graphs of the eigenvalues, associated with selected eigenfunctions, versus $a/r = a$ and examined consecutive pairs of curves.
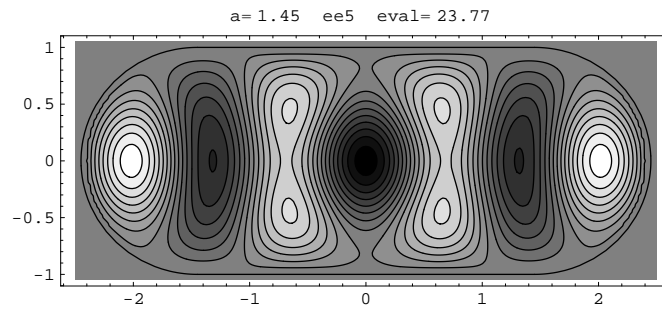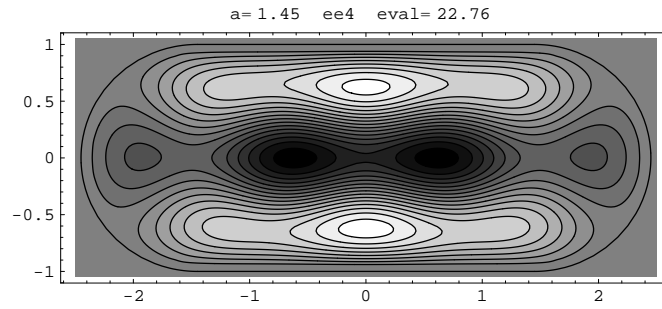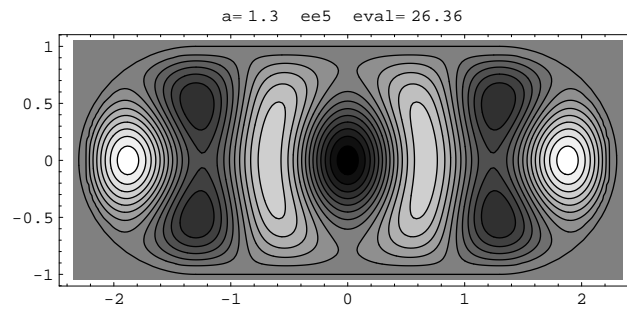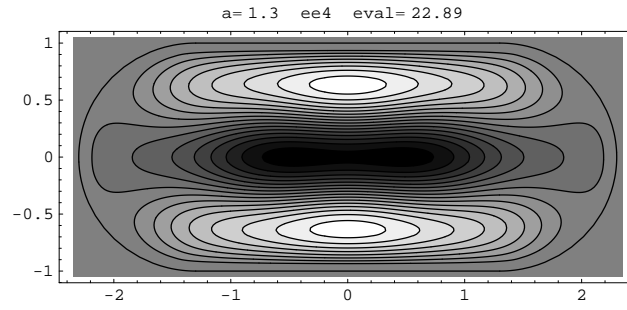
The first figure indicates the third, fourth and fifth eigenvalues corresponding to the eigenfunctions of even-even symmetry over a series of values a in $[0, 2]$. We denote by f_3, f_4, f_5 the simple curves of those eigenvalueswith respect to a. The avoided crossings between the pair (f_3, f_4) at approximately .785 and the pair (f_4, f_5), at about 1.51, are evident visually. If one half closes one's eyes it appears that the pairs cross. After a closer observation it then appears that the upper curve of each pair continues in the same direction its partner was taking before the avoidance, and vice versa.
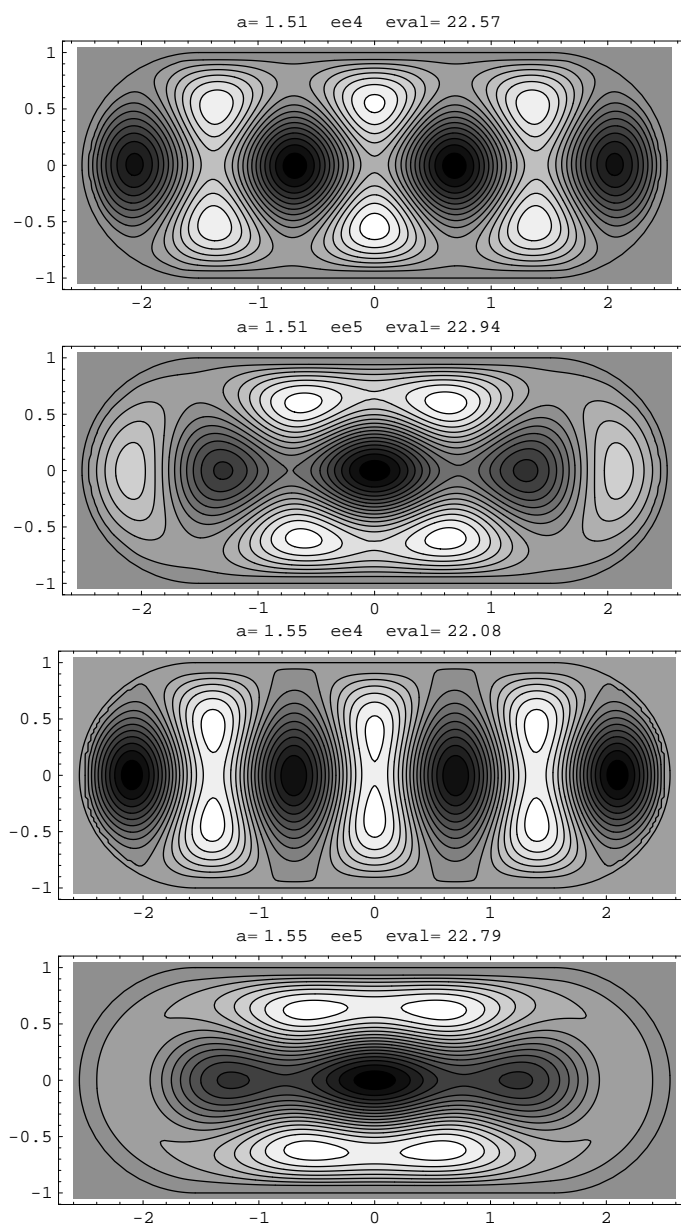
The idea of the work here is to examine the eigenfunctions that correlate with the line curves with particular attention given to contour plots near the avoided crossings to observe how sensitive the data is to the separation values a. The corresponding pairs of contour plots ($ee3, ee4, ee5$) seem to 'swap' characteristics just as the curve pairs tended to take over each other's directions.
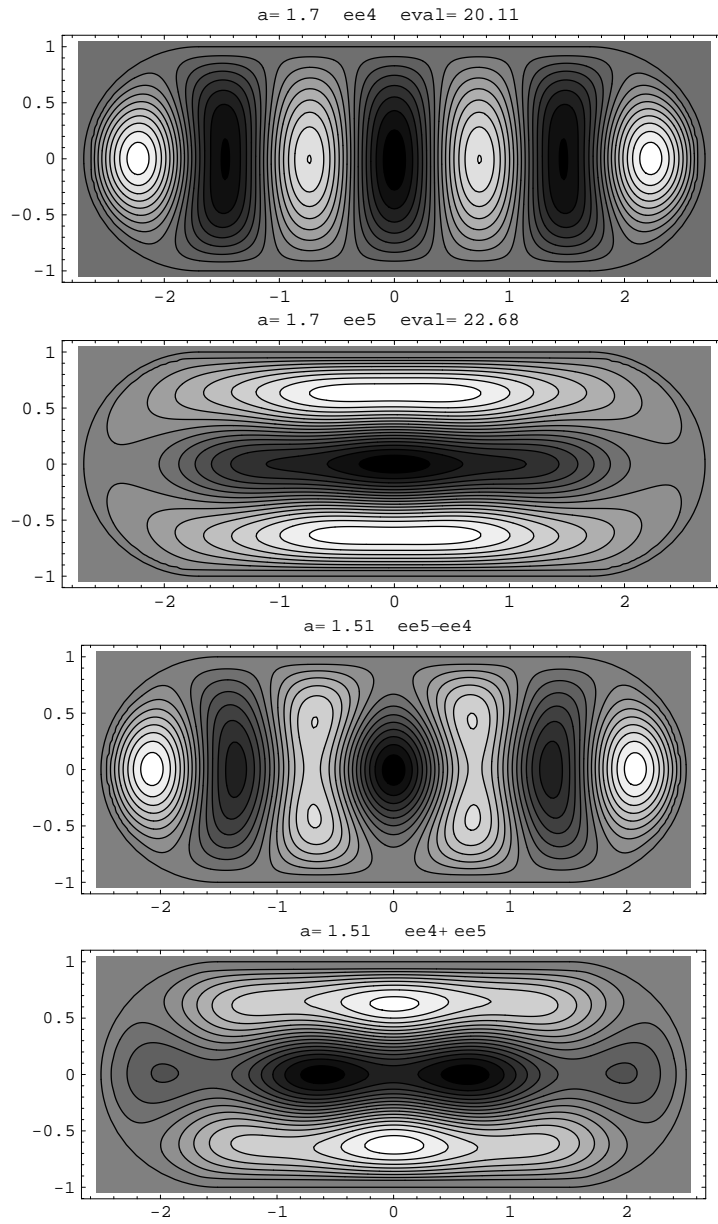
Jumping across the narrow gap between f_4 and f_5 the *difference* between the two eigenfunctions demonstrates a graphical affinity for the subsequent characteristics of $ee5$ and the previous characteristics of $ee4$, while the *sum* goes the other way (last two contour graphs). In this paper we are not drawing any physical conclusions but submit some of our graphical results for the reader's inspection.

a= 1.3   ee4   eval= 22.89

a= 1.3   ee5   eval= 26.36

a= 1.45   ee4   eval= 22.76

a= 1.45   ee5   eval= 23.77

a= 1.51   ee4   eval= 22.57

a= 1.51   ee5   eval= 22.94

a= 1.55   ee4   eval= 22.08

a= 1.55   ee5   eval= 22.79

a= 1.7   ee4   eval= 20.11

a= 1.7   ee5   eval= 22.68

a= 1.51   ee5−ee4

a= 1.51   ee4+ ee5

References

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